Dynamic Coupling and Market Instability

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Abstract

We examine dynamic coupling and feedback effects between High Frequency Traders (HFTs) and how they can destabilize markets. We develop a general framework for modelling dynamic interaction based on recurrence relations, and use this to show how unexpected latency and feedback can trigger oscillatory instability between HFT market makers with inventory constraints. Our analysis suggests that the modelled instability is an unintentional emergent behaviour of the market that does not depend on the complexity of HFT strategies — even apparently stable strategies are vulnerable. Feedback instability can lead to substantial movements in market prices such as price spikes and crashes.

Keywords: HFT, market maker, latency, feedback, instability, coupling

2000 MSC: C61, 63, G19, G14

1. Introduction

In a recent journal special issue on High Frequency Trading (HFT) Chordia et al highlighted a key unanswered problem: what is “the nature of the mechanism by which the interaction of HFT algorithms improves market quality”? (Chordia et al., 2013). In the same issue, Hasbrouck and Saar said they “cannot rule out that in times of severe market conditions HFTs may contribute to market failure” (Hasbrouck and Saar, 2013). Here, we contribute to this debate by investigating the low-level mechanisms by which
High Frequency Traders (HFTs) may interact to reduce market quality and lead to failure, especially during times of market disequilibrium. We show how coupling and feedback loops may occur between HFTs, we introduce a general framework for modelling dynamic interaction between financial algorithms, and we show how latency and feedback loops may trigger instability as an unintentional emergent behaviour of the market.

Concern has previously been expressed about the potential for feedback loops to impact prices and destabilize markets (Danielsson et al., 2012; Zigrand et al., 2012). Feedback loops are not only widespread within the financial markets but may also exist for a long time, in some cases possibly remaining unnoticed. The adverse effects of a destabilising feedback loop may only become apparent when its strength becomes sufficiently large.

A prominent example of market instability either arising from or exacerbated by HFTs was the Flash Crash of May 6th 2010 (CFTC-SEC, 2010). Of direct relevance to our work is the “hot potato” trading behaviour of market makers at the heart of the Flash Crash, where multiple HFTs traded with each other in a rapid oscillation of large aggressive orders. This highly unusual oscillatory instability created both deceptive trading volume (which implied liquidity where none was present) and a spike in messaging traffic that stressed the already-overloaded technology infrastructure (Nanex, 2010c).

We model dynamic coupling and feedback between HFTs at the level of the market microstructure, and expose the underlying mechanics of dynamic interaction. We illustrate this with a case study of interaction between inventory-driven HFT market makers (Menkveld, 2013) in an order-book market, each executing a simple, stable trading strategy. Unlike those observed by (Hasbrouck and Saar, 2013), our HFTs do not intentionally interact or “play” with each other. Nevertheless, we explain how dynamic coupling between our HFTs leads to a feedback loop where each HFT influences the behaviour of the other, and how this feedback has the potential to generate unintentional instability (including highly volatile oscillatory behaviour) as an emergent behaviour of the market.

Our model shows that one of the triggers for such behaviour is the introduction of unexpected additional latency (unexpected delay), as might be experienced for example when a sudden burst of quotes overwhelms an execution venue and causes market data feeds to be delayed. We illustrate how we model and analyse a market where traders experience unexpected delay and how this leads to instability.

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The primary contribution of this article is to provide an alternative narrative for market instability, using our dynamic interaction model to show in great detail how latency and feedback loops between dynamically-coupled HFTs may trigger unintentional market instability. Although our case study focuses on oscillation arising from the interaction of automated market-making strategies, we suspect that many of the previously observed feedback loops (e.g. in Danielsson et al. (2012)) and the impact of market maker inventories on time-varying liquidity (Comerton-Forde et al., 2010) may also be modelled and analysed using the techniques we describe in this article. Our work may also have implications for models of pricing and market impact, since we demonstrate that traders do not necessarily have independence of action and such models may need to account for unexpected coupling with other traders.

The paper is organized as follows. First we set out the relationship with prior work, followed by an introduction to modelling coupling and feedback. Section 4 details our dynamic interaction model for a simple case study, and Section 5 analyses this model and makes the link from coupling and feedback to market instability (including numerical simulation results, which illustrate some theoretically infinite modes of oscillation). Section 6 concludes, and is followed by an appendix containing further definitions for our case-study.

2. Relation to prior work

Previous theoretical and empirical studies of instability have predominantly focused on interaction as an indirect process via prices (Arthur et al., 1996; Caldarelli et al., 1997), or via globally-shared information (Brock and Hommes, 1998; Lux and Marchesi, 1999; Hommes and Wagener, 2009) without providing a detailed exploration of the underlying mechanism of interaction that causes such price fluctuations. Where direct interaction is included in the model, it is often abstracted — for example, using an Ising model to provide an abstraction of nearest-neighbour communication between traders (Kaizoji, 2000; Iori, 2002), or assuming traders form bidirectional links in a random network (Cont and Bouchaud, 2000). By contrast, we explore the mechanistic order-by-order interaction within the limit order book, where we explicitly model multiple bilateral direct interactions between traders.

We follow Iori (2002) and Cvitanic and Kirilenko (2010) by modelling dynamic interaction in discrete time, thereby exposing substantial microstructure detail such as the discrete nature of computer messaging, of order pro-
cessing by heterogeneous traders, and the discrete nature of order arrival and order processing by the limit order book (Day and Huang, 1990).

We have found recurrence relations to be the most helpful technique for our discrete-time models. Recurrence relations have not previously been used to express coupling dependencies, but they have been used in related work such as price instability caused by fundamentalist/chartist interaction (Chiarella et al., 2006), clustered volatility caused by feedback effects (Farmer and Joshi, 2002), price instability from time-varying demand (Day and Huang, 1990), and instability from leveraging (Thurner et al., 2012).

Game-theoretic models focus on the systemic instability effects of interaction between traders (Giardina and Bouchaud, 2003; Brunnermeier and Pedersen, 2005), yet they start from the premise that instability arises from the complexity of traders adapting the value of some internal parameter in order to optimise a utility function, and the model is used to explore market equilibria. By contrast, we are interested in systemic effects that arise from traders with fixed strategies where no optimisation is involved and we explore market disequilibria and detailed causation at a microstructure level.

Other studies of feedback in financial markets include: Gennotte and Leland (1990), who model price instability and show how the extent of a price crash may be determined by the feedback effect arising from unobserved portfolio hedging; Bouchaud and Cont (1998), who emphasize the role of feedback effects in market instability (in particular through risk aversion) and utilise a Langevin equation to model feedback mediated via prices; and Westerhoff (2003), whose agent-based numerical simulation explores risk-averse market making strategies in foreign exchange markets and shows how feedback interaction between market makers and speculators can increase trading volume and distort exchange rates.

Menkveld and Zoican (2014) have recently provided a Markovian recursive model of interaction between HFT market makers and predators; this is in the same spirit as our model, in that it models direct interaction in discrete time and aims to “uncover effects that remain hidden in static models” — however, we develop a more general framework that for example supports a full order book, dependencies on historical values, independent communication delays, and the tracking of multiple variables at each time step.

Our work extends the current understanding of interaction-based instability by examining the detailed mechanisms (including latency and feedback) by which HFTs (and others) can become dynamically coupled, causing them to operate unintentionally as a collaborative unit that leads to nonlinear
oscillation and unstable markets. We suspect dynamic coupling may be implicated in a range of previously observed instabilities.

3. Background to modelling feedback loops

For our purpose of modelling feedback loops, we consider a market to be comprised of multiple subsystems, which may overlap. The smallest subsystem is a single component; components can be any entity — for example, a human trader or a trading algorithm or a news source, though components may be larger (an exchange) or smaller (a risk management subroutine). A subsystem may contain components or further subsystems. What is important for the model is the interaction between these subsystems.

3.1. Coupling

We say that if the behaviour of one subsystem influences the behaviour of another the latter is coupled to the former and the two comprise a larger system that exhibits coupling. For example, two HFTs are coupled if one mimics the operations of the other. The HFTs would also be coupled if one of them acted according to some pattern triggered by the other’s activity. Any two subsystems may be coupled and there may exist chains of coupled subsystems (e.g. one HFT is coupled to another that in turn is coupled to a news source). If a subsystem is coupled to itself either directly or indirectly via a coupling chain, we define this to be a feedback loop and all the subsystems in this cyclic chain are said to be mutually coupled.

We say a coupling is static if it always persists, with constant strength: a coupling is dynamic if it is transient or has varying strength. Dynamic coupling is less predictable, hence more dangerous, than static coupling. We define market instability as a large change or volatility in one or more market parameters such as market price, traded volume, or frequency of trading.

3.2. Oscillation, phases and phase-shifting

Oscillation can arise in many ways: for example, the interaction between momentum and fundamental traders can lead to oscillatory price behaviour (Sethi, 1996; Chiarella et al., 2006). Oscillatory instability may also arise from two stabilising feedback loops. Consider a simple contrarian trader who buys when prices are too low, and sells when prices are too high. Assume the prices of the trader’s limit orders are set to be halfway between the current market price and a reference price ($T_b$ for buying and $T_s$ for selling); thus the
trader’s behaviour is *coupled* to the market price. If these orders affect the market price, causing the price to drop when selling and to rise when buying, then the market price is also *coupled* to the trader behaviour. A two-way coupling is created and each phase (buying/selling) creates a stabilising feedback: when selling, market price asymptotically drops to \( T_s \); when buying, market price asymptotically rises to \( T_b \). Now if \( T_b > T_s \) and if the strategy exhibits inertia between \( T_b \) and \( T_s \) (if it was previously buying(selling), it continues until \( T_b(T_s) \), at which point it switches to selling(buying)), the combined effect is an oscillatory instability as illustrated in Figure 1.

![Diagram of Oscillatory Instability](image)

**Figure 1:** Oscillatory instability arising from two stabilising feedback loops: (a) the behaviours of the two trader feedback loops (the lines plot price during trader selling phase and buying phase); (b) the three phases for the market as a whole — price bands identified for rising (\( price < T_s \)), oscillating (\( T_s \leq price \leq T_b \)) and falling (\( price > T_b \)) prices.

Modelling such a market is complex because the behaviour of the market in the region \( T_b > P > T_s \) is not entirely determined by the price \( P \) but also by the current behavioural phase of the trading algorithm. The trader in this example displays two behavioural *phases*: buying and selling. We also say that the overall market exhibits three phases: a falling-price phase (\( P \geq T_b \)), a rising-price phase (\( P \leq T_s \)), and an oscillating-price phase (\( T_b > P > T_s \)).

More precisely, we say that a *phase* of a subsystem (or of a market) is a set of measurable properties which define a distinctly different behaviour of the subsystem, and the transition from one phase to another we call *phase-shifting*. Phase-shifting by itself is not necessarily a sign of instability, however we shall show later how phase-shifting can lead to instability.

### 4. The Model

In this section we explain how we model dynamic coupling and feedback in a financial market. We start with some background comments to explain what we wish to model and why we adopt a particular approach; then we
illustrate our method with a case study. Our initial model is a market with two market makers and an exchange; we then add a fundamental seller, and additional unexpected communication latency (delay).

4.1. Background to the model

Our aim is to model coupling and feedback at a sufficient level of detail to investigate how they arise, how they operate, and how they contribute to market instability. Our technique is deterministic rather than probabilistic, in order to expose precise mechanistic causality. For example, the precise timing and interleaving of order flow may be critical to the analysis of instability arising from HFT interactions.

In general, when modelling a market with complex feedback loops, the subsystem behaviours of interest may not be expressible in analytic form, they may depend on local memory (e.g. whether to buy or sell at a price may depend on whether the price has recently been rising or falling), they may be rugged (non-smooth), and they may exhibit complex inter-relationships.

Here we present our deterministic discrete-time model using mutually-recursive recurrence relations. A key characteristic of our model is that it supports the direct expression of coupling in the structure of the model. The recurrence relations describe how the value of a subsystem parameter changes with time. Where one relation references another, this indicates a dependency or coupling; where the latter also references the former, this indicates a simple feedback loop. For example, consider the following relations for parameters $X_t$, $Y_t$ and $Z_t$ (for general functions $f()$, $g()$ and $h()$):

$$
X_t = f(Y_{t-1}) \quad Y_t = g(Z_{t-1}) \quad Z_t = h(Y_{t-1})
$$

In the above equations, $X$ is unidirectionally coupled to $Y$ and there is a bidirectional coupling between $Y$ and $Z$. In a slightly less abstract example of feedback, consider traders issuing sell orders based on a delta-hedging model: the delta-hedging model depends on the price of a stock index; the index depends on the prices of the component stocks; and the prices of stocks depends on the price impact of sell orders arriving from the traders:

$$
sellorders_t = \text{deltahedge}(\text{index}_{t-1})
$$

$$
marketprice_t = \text{priceimpact}(\text{marketprice}_{t-1}, sellorders_{t-1})
$$

$$
\text{index}_t = \text{indeximpact}(\text{marketprice}_{t-1})
$$

\footnote{The functions $f()$, $g()$ and $h()$ might express linear or nonlinear couplings.}
It is sometimes more convenient to use a single function to define the behaviour of two or more parameters simultaneously. For example, parameters $R_t$ and $S_t$ may be defined as the result of some general function $f()$:

$$ (R_t, S_t) = f(R_{(t-1)}, S_{(t-1)}) $$

Where the above style of description is used, the coupling is made explicit inside the function $f()$ (for example, Equation (4) in Section 4.2.1 uses a function $\text{match}()$ whose definition contains the detailed couplings).

We may also wish to explore the behaviour in time of a market parameter $X$; this is achieved by animating the aforementioned equations from some starting values (e.g. $X_0, Y_0$) and plotting the sequence $\{X_0, X_1, \ldots X_{\text{final}}\}$. This provides a time series such as that shown in Figure 1.

4.2. Case Study

To illustrate our technique, we provide as a case study the detailed model of dynamic coupling with oscillatory feedback between risk-averse HFT market makers. Although HFT market makers typically straddle multiple exchanges (Menkveld, 2013), our case study comprises a single exchange, with a number $\text{maxi}$ of market makers each identified by a number $i$; we then add a fundamental seller. In our model the exchange manages two limit order books — $\text{bidbook}_t$ for bids, and $\text{askbook}_t$ for asks.

Our method permits order flow to be modelled with each trader issuing one order at each time step, or many orders. In our case study, each trader issues up to four orders per time step (one bid, one ask, one sell and one buy), and the exchange processes all orders received at one time step before considering orders received at the next step.

4.2.1. Couplings

Figure 2 illustrates the couplings for a minimal system with an exchange and two HFT market makers. Although this is a simplified version of real market couplings, it is still complex. Many arrows in Figure 2 represent dynamic couplings; e.g. a sell order is determined by the difference between a trader’s inventory and its inventory threshold, and this varies with time.

\footnote{Bids and asks are resting (passive) orders. Buys and sells are aggressive executable orders; e.g. immediately executable limit orders or market orders.}
Such feedback diagrams are not easy to analyse. We provide a formal, and tractable, description by introducing a discrete time dimension. Each time \( t \) represents the point when a message is sent from one entity to another, and the step from \( t \) to \( t + 1 \) represents the time taken for an entity to receive incoming data, process it, and issue a new message. Many couplings occur within a time-step but others are time-dependent where the behaviour of one entity is coupled to the behaviour at a previous time of another entity (in Figure 2 these are arrows with ovals at their tails). Our model requires that every cyclic chain of couplings includes at least one time-dependent coupling.

In the model, all communication occurs synchronously — either all entities are processing, or they are all sending and receiving messages. Nevertheless, it is possible to model entities that take differing amounts of time to calculate what to do next (e.g. a slow trader), and it is also possible to model traders issuing orders at different times. Both of these effects are achieved by supporting “empty” messages; for example, a fast trader might send orders every even timestep and send an “empty” message every odd time step, and a slow trader might send orders only on every tenth timestep (otherwise it sends empty messages). Thus, we can specify the relative speeds of different subsystems. Furthermore, we shall see in Section 4.3 how specific delay components can be used to model different latencies in communications links.

For simplicity, we assume all traders and the exchange take the same time to process incoming data and issue messages. Thus, the total round-trip time between a trader and the exchange is two time steps, and if a trader waits for a response to one order before issuing the next then that trader will only issue orders on alternate time steps. We hereafter assume that traders issue orders on even time steps and the exchange issues confirmations on odd steps.

The value of a parameter at time \( t \) may depend on its own previous value at time \( t - 1 \) (or at any previous time, but it may not depend on its value at the same time \( t \)). This is an example of a time-dependent coupling. For example, we say that the inventory for trader \( i \) at time \( t + 1 \) (denoted by \( inv_{i,(t+1)} \) ) is coupled both to its value at the previous time step \( inv_{i,t} \) and to the sizes of the confirmed executions (sent from the exchange at time \( t \) ) of that trader’s previously issued orders: \( x\text{bids}_{i,t} \), \( x\text{asks}_{i,t} \), \( x\text{buys}_{i,t} \) and \( x\text{sells}_{i,t} \).

Figure 2 denotes the dependency on the sizes of the executed orders as a time-dependent feedback dependency (coloured black) because the inventory depends on the sizes of the executions, which depend on previously issued...
orders, which in turn depend on previous inventory.  

We introduce the selection function $\psi(i, x)$ (see Appendix A) to sum the sizes of all (and only) those orders issued by trader $i$ in a set of orders $x$, and our discrete-time recurrence relation for the inventory for market maker $i$ is:

$$inv_{i,(t+1)} = \begin{cases} 
inv_{i,t} + \psi(i, xbids_t) + \psi(i, xbuys_t) \\
- \psi(i, xasks_t) - \psi(i, xsells_t)
\end{cases}$$

(1)

Equation (1) holds at all time steps, but inventory will only change on even steps since the confirmations ($xbids$ etc) are only issued on odd steps.

Similarly, we define the couplings (to current inventory $inv_{i,(t+1)}$) that determine the size of each market maker’s sell and buy orders. We introduce the functions $buysize()$ and $sellsize()$ (see Section 4.2.2), which embody the market-maker internal logic for determining buy and sell sizes, and the function $order()$, which takes an order type, size, price and identifier, and returns an order. Orders are only issued on even time steps:

$$buy_{i,(t+1)} = order(buy, buysize(inv_{i,(t+1)}), \nu, i)$$
$$sell_{i,(t+1)} = order(sell, sellsize(inv_{i,(t+1)}), \nu, i)$$

We also introduce the functions $bidsize()$ and $asksize()$ (see Section 4.2.2) to embody the sizing logic for resting limit order (coupled to inventory), and the functions $bidprice()$ and $askprice()$ (see below) to embody the pricing logic (coupled to both inventory and order-book information). This defines how limit orders are coupled to trader inventory and order-book information as illustrated in Figure 2, where the dependency on order-book information is coloured black to denote a feedback dependency (e.g. because the best bid price depends on the previously issued bids and the previously issued bids depended on the previous best bid price). Again, for even time steps only:

$$bid_{i,(t+1)} = order(bid, bidsize(inv_{i,(t+1)}), bidprice(bestbid_t, bestask_t, inv_{i,(t+1)}), i)$$
$$ask_{i,(t+1)} = order(ask, asksize(inv_{i,(t+1)}), askprice(bestbid_t, bestask_t, inv_{i,(t+1)}), i)$$

---

3 Although feedback couplings are generally time-dependent, time-dependent couplings need not be feedback couplings.

4 $\nu$ is the price of the executable order — for the rest of this paper, we assume executable orders are market orders with no price (represented by $\nu=0$).
Amihud and Mendelson (1980), Comerton-Forde et al. (2010) and Menkveld (2013) show how market makers skew order prices to control their inventories, and in our case study we use a very simple version of this behaviour: we set limit order prices such that for high inventory both bid and ask prices are low (encouraging more asks and fewer bids to be executed) and for low inventory both bid and ask prices are high (encouraging more bids and fewer asks to be executed). We ensure that bid and ask prices are not negative, new bid prices are not higher than \( \text{midprice} - 1 \), and new ask prices are not lower than \( \text{midprice} + 1 \) (so resting orders are never crossed).
Figure 2: Coupling between components for a market with one exchange Exchange and two HFT market makers Trader$_1$ and Trader$_2$. Arrows are unidirectional dependencies (the head is coupled the tail): bold black arrows are key feedback dependencies (the dashed arrows are explained in Section 4.2.1). If an arrow's tail has an oval it is a time-related dependency — the head is coupled to the value of the tail at a previous time. Traders may issue sell or buy orders (determined only by current inventory inv) and may also issue bid or ask orders (determined by inventory inv and knowledge of the best bid and best ask at the exchange). Orders are grouped (e.g. Bids) before being added to the intermediate bidbook' and askbook' and matched to produce sets of executed orders (e.g. \{xbids$_i$\}, \{xsells$_i$\}) and the resulting bidbook and askbook. Each trader only sees his/her own confirmed executions. Rectangles indicate functions that are described in the text. In this simple model, crossed bids and asks are not executed against each other; neither are buys against sells.
Thus in our model the pricing functions have the simple form:\textsuperscript{5}
\begin{align*}
\text{bidprice}(\text{bestbid}, \text{bestask}, \text{inv}) &= \max(0, (\text{midprice} - 1) - \alpha \times \text{inv}) \\
\text{askprice}(\text{bestbid}, \text{bestask}, \text{inv}) &= \max(0, (\text{midprice} + 1) + \beta \times \text{inv})
\end{align*}
(2)

The orders from the two market makers are grouped before being processed by the exchange and we represent these groups as ordered sequences using the notation \{...\} (where \{\} is the empty sequence):

\begin{align*}
\text{Bids}_{(t+2)} &= \{\text{bid}_{1, (t+1)} \ldots \text{bid}_{\text{maxi}, (t+1)}\} \\
\text{Asks}_{(t+2)} &= \{\text{ask}_{1, (t+1)} \ldots \text{ask}_{\text{maxi}, (t+1)}\} \\
\text{Buys}_{(t+2)} &= \{\text{buy}_{1, (t+1)} \ldots \text{buy}_{\text{maxi}, (t+1)}\} \\
\text{Sells}_{(t+2)} &= \{\text{sell}_{1, (t+1)} \ldots \text{sell}_{\text{maxi}, (t+1)}\}
\end{align*}

The exchange then processes the incoming orders.\textsuperscript{6} Bids and asks are added to the \textit{bidbook} and the \textit{askbook} to create intermediate books \textit{bidbook}' and \textit{askbook}'. In our model the order books are ordered sequences of limit orders and the positions of the limit orders within an order book are determined by their price and their time of arrival.\textsuperscript{7} The first bid (ask) in \textit{bidbook}' (\textit{askbook}') is the one with the highest (lowest) price (and where there is more than one bid (ask) at that price, they are sorted in order of arrival time so that the earliest arriving bid (ask) is the first in the sequence). We introduce the further notational device \textit{x : y} to represent a sequence of orders where the first order in the sequence is \textit{x} and \textit{y} is the remainder of the sequence with \textit{x} removed. It follows from the above that if \textit{bidbook} = \textit{b : bs} then the best bid is \textit{b}, and if \textit{askbook} = \textit{a : as} then the best ask is \textit{a}.

To add new orders to an order book, we introduce the functions \textit{insertbid()} and \textit{insertask()} (defined in Appendix A). If we wished bids to rest on the bidbook until cancelled, we would define \textit{bidbook}' as:

\[
\text{bidbook}'_{(t+2)} = \text{insertbid(\text{bidbook}_{(t+1)}, \text{Bids}_{(t+2)})}
\]

However, to simplify the presentation of our case study, we assume that all orders are Fill And Kill (they are fully or partially executed immediately or

\textsuperscript{5} Detailed expressions for \textit{\alpha} and \textit{\beta} are given in Appendix A but are not necessary to understand this presentation.

\textsuperscript{6} This simple model assumes all orders are guaranteed to be delivered to and accepted by the exchange, though it is also possible to model order confirmations in the case of a system where one or both of these assumptions does not hold.

\textsuperscript{7} The coupling of \textit{bidbook} and \textit{askbook} to order arrival time is not shown in Figure 2.
otherwise cancelled — also known as Immediate Or Cancel). We therefore empty the order books before adding newly arrived orders:

$$
\begin{align*}
\text{bidbook}'_{t+2} &= \text{insertbid}(\emptyset, \text{Bids}_{t+2}) \\
\text{askbook}'_{t+2} &= \text{insertask}(\emptyset, \text{Asks}_{t+2})
\end{align*}
$$

(3)

The exchange then matches the incoming sell (or buy) orders issued at time \( t + 1 \) with those limit orders resting on the \( \text{bidbook}'_{t+2} \) (or \( \text{askbook}'_{t+2} \)) to determine the new trade executions \( \text{xbids}_{t+2} \) and \( \text{xells}_{t+2} \) (or \( \text{xasks}_{t+2} \) and \( \text{xbuys}_{t+2} \)) and the new \( \text{bidbook}_{t+2} \) (or \( \text{askbook}_{t+2} \)). The dependency of \( \text{bidbook}_{t+2} \) (\( \text{askbook}_{t+2} \)) on \( \text{bidbook}_{t+1} \) (\( \text{askbook}_{t+1} \)) is another example of a benign feedback and is coloured black in Figure 2.

The exchange’s matching engine is represented by the function \( \text{match}() \), which must also remove executed limit orders from the relevant order book (discussed below). Confirmations are only issued on odd time steps:

$$
\begin{align*}
(\text{bidbook}_{t+2}, \text{xbids}_{t+2}, \text{xells}_{t+2}) &= \text{match}(\text{bidbook}'_{t+2}, \text{Sells}_{t+2}) \\
(\text{askbook}_{t+2}, \text{xasks}_{t+2}, \text{xbuys}_{t+2}) &= \text{match}(\text{askbook}'_{t+2}, \text{Buys}_{t+2})
\end{align*}
$$

(4)

Each executed sell (buy) will be at the price of the currently best bid (ask); and if the size of the sell (buy) is greater than that of the best bid (ask), this may change the subsequent best bid (ask) price used for the next execution. Thus, the executed sell (buy) prices are coupled to (i) the current best bid (ask) prices, (ii) the sizes of the executed sell (buy) orders, and (iii) the sizes of the bids (asks) in \( \text{bidbook}' \) (\( \text{askbook}' \)).

The operation of the matching engine is complex (see Appendix A), but we can express the executed sell prices and buy prices inductively using the function \( P(r : rr, e : ee) \) where \( e : ee \) is a sequence of executable orders to be matched against an ordered sequence \( r : rr \) of resting limit orders (using the notation introduced above). We consider one executable order at a time; we match each executable order against the resting limit orders on the relevant order book — if there is no liquidity, there are no more trades, but otherwise we have (where \( \pi() \) gives the price of an order, \( \sigma() \) gives the size of an order, and \( \rho(x, y) \) reduces the size of \( x \) by the size of \( y \)):

$$
P(r : rr, e : ee) = \left\{ \begin{array}{ll}
\pi(r) : P(\rho(r, e) : rr, ee) & \text{if } (\sigma(e) < \sigma(r)) \\
\pi(r) : P(rr, \rho(e, r) : ee) & \text{if } (\sigma(e) > \sigma(r)) \\
\pi(r) : P(rr, ee) & \text{if } (\sigma(e) = \sigma(r))
\end{array} \right.
$$

(5)

The above equation for market prices specifies exactly how the matching engine “walks the book” in order to fill an executable order — first executing
against the best-priced resting limit order and, if that was insufficient to fill the executable order, progressing to the next-best resting order. Given a particular distribution of limit orders on the book, a large total size of executable orders is more likely (than a small total size) to deplete the top price level on the relevant order book and cause a jump in the execution price (see Figure 3). Whether a price change will occur at all is simply given by comparing the total sizes of all executable orders (in either Sell or Buy) with the total sizes of all the resting limit orders resting on the relevant book at the best price (if the former is greater, then at least one execution will be at a different price). How much the price moves will depend on the distribution of limit orders on the book — especially the distribution near the top of the book, and the degree of “gapping” in that distribution.

![Figure 3: Matching sells against bids: the executed prices occur first at the best bid (a); then at (b) since this will have become the new best bid; and finally a larger price jump to (c). The bolded portions of the bid lines are the executions.](image)

The information from bidbook_t and askbook_t is published by the exchange, including the best bid and ask prices which are used by the traders in calculating the next set of limit orders. The values x_bids_t, x_asks_t, x_buys_t and x_sells_t are the local trade confirmations sent by the exchange to the relevant traders (the two counterparties to the trade). Finally, Figure 2 contains two bold dashed black arrows which merge with two other feedback arrows — this represents an optional price banding constraint that might be applied as a market protection mechanism.\(^8\)

This completes our set of recurrence relations to model the main dependencies illustrated in Figure 2. We note in particular that the existence of

\(^8\)For a practical example of price banding, see [http://www.cmegroup.com/confluence/display/EPICSANDBOX/GCC+Price+Banding](http://www.cmegroup.com/confluence/display/EPICSANDBOX/GCC+Price+Banding)
identified feedback dependencies makes it possible to trace many feedback
loops — including feedback loops that involve both market makers, as can be seen from the feedback loops crossing the horizontal midline in Figure 2.

4.2.2. Phase shifting

The previous section described how we model dependencies between system components, but we have not yet described the detailed behaviour of the market-making algorithm. In particular, we wish to model the phase-shifting of an algorithm between two different types of behaviour. Computer trading algorithms are frequently subject to phase-shifting, typically implemented as conditional branches to choose between different behaviours in different market contexts. Here we create an algorithm with a somewhat simplistic shift between two dramatically different behaviours — in practice, an algorithm might exhibit many phases and the switching between phases might be more subtle than this example.

We model a simple risk-averse, long-short market maker that actively manages risk based on the size of the current inventory (Manaster and Mann, 1996). Although in practice a market maker could make complex risk calculations, it suffices for our model simply to use raw inventory (since we are only interested in the switching between behaviours and not precise values). Our market maker uses a threshold policy (Huang et al., 2012) with an upper-bound inventory limit $UL$ and a lower-bound inventory limit $LL$ (a negative number). To simplify the presentation, these limits are assumed to be fixed, though in practice they could vary according to market risk factors such as observed volatility. Based on these inventory limits, our risk-averse market maker phase-shifts between two different behaviours:

1. A stable phase whenever $LL < inv_{i,t} < UL$, where only resting limit orders are issued — at each even time step (to allow for round-trip communication with the exchange) both a bid and an ask are issued. We assume that all orders are Fill And Kill.\textsuperscript{9} A special situation arises for $inv_{i,t} = LL+1$ where only a bid is issued and $inv_{i,t} = UL-1$ where only an ask is issued.

2. A panic phase whenever $inv_{i,t} \geq UL$ or $inv_{i,t} \leq LL$, where at each even time step either a large sell or a large buy is issued in an attempt to

\textsuperscript{9}See (Chakraborty and Kearns, 2011) for a similar model, though in our simple case study we only place one bid and one ask at each time step.
revert inventory to zero. In our model, executable order size is either $UL$ for a positive-inventory panic or $-LL$ for a negative-inventory panic — in practice, the size might also depend on market conditions and constraints, but we find this simple approximation is sufficient for our initial model. No resting limit orders are issued in a panic phase.

Phase-switching is defined in the functions that determine order size:

$$\begin{align*}
\text{buysize}(inv_{i,t}) &= \begin{cases} 0 & \text{if } (inv_{i,t} > LL) \\ -LL & \text{otherwise} \end{cases} \\
\text{sellsize}(inv_{i,t}) &= \begin{cases} 0 & \text{if } (inv_{i,t} < UL) \\ UL & \text{otherwise} \end{cases}
\end{align*}$$

$$\begin{align*}
\text{bidsize}(inv_{i,t}) &= \begin{cases} 0 & \text{if } (inv_{i,t} \geq UL) \\ \text{bidsize}'(inv_{i,t}) & \text{otherwise} \end{cases} \\
\text{asksize}(inv_{i,t}) &= \begin{cases} 0 & \text{if } (inv_{i,t} \leq LL) \\ \text{asksize}'(inv_{i,t}) & \text{otherwise} \end{cases}
\end{align*}$$

Precise limit order size is delegated to the functions $\text{bidsize}'()$ and $\text{asksize}'()$. Our model does not require any particular values to be chosen, but we observe that if the limit order sizes are chosen to be within the shaded region of Figure 4 then under normal circumstances the market maker will not switch into a panic phase if it starts in the stable phase (see Section 5.2). For our case study, we set bid and ask sizes to be exactly the maximum that will never exceed the inventory limits $UL$ and $LL$ (to keep the presentation simple, we ignore details such as minimum size constraints imposed by the exchange, and we use a single large order rather than splitting into several smaller orders). Thus:

$$\begin{align*}
\text{bidsize}'(inv) &= \max(0, UL - 1 - inv) \\
\text{asksize}'(inv) &= \max(0, inv - (LL + 1))
\end{align*}$$

This aligns (somewhat simplistically) with the empirical observation of (Kirilenko et al., 2010) that “HFTs do not accumulate a significant net position and their position tends to quickly revert to a mean of about zero”, and with the Nanex description of HFT behaviour during the Flash Crash: “they slammed the market with 2,000 or more contracts as fast as they could” (Nanex, 2010b).
Figure 4: Risk-averse order sizes as a function of inventory, given inventory limits $LL$ and $UL$. Bidsize crosses the y axis at $UL - 1$ and asksize crosses the y axis at $-(LL + 1)$. If bid (ask) size is chosen within the shaded area beneath the “bidsize” (“asksise”) line, then it is impossible under normal conditions for inventory to reach the limit (see Section 5.2).

4.2.3. Adding a fundamental seller

In the next section we will require a fundamental seller to provide sell orders to trade with the market makers. We therefore define the equations for a trader with index 0 whose behaviour is to issue a sell order of a fixed size $\omega$ at every even time step (it does nothing else) up to a predetermined time $timelimit$, and then exits the market. Thus (for even time steps only):

\[
\begin{align*}
buy_{0,t} &= order(buy, 0, 0, 0) \\
sell_{0,t} &= \begin{cases} 
\text{order}(\text{sell}, \omega, 0, 0) & \text{if } (t < timelimit) \\
\text{order}(\text{sell}, 0, 0, 0) & \text{otherwise}
\end{cases} \\
bid_{0,t} &= order(bid, 0, 0, 0) \\
ask_{0,t} &= order(ask, 0, 0, 0)
\end{align*}
\]

4.3. Modelling information delay

Information delay is a known and widespread source of instability in the financial markets (Beja and Goldman, 1980; Chiarella, 1992; CFTC-SEC, 2010; Tse et al., 2012). We define information delay to be an unexpected additional latency in transmitting information; it may manifest in different ways throughout a financial market, and may affect all kinds of information. For example: (i) delays in financial and economic news, and consequent delays in relevant information being incorporated into traded prices (Beja and Goldman, 1980; Chiarella, 1992); (ii) delays due to exchange throttling; (iii) delays due to technology infrastructure having switched to a business-
continuity site; (iv) delays in market data: both direct feeds (Nanex, 2010c; Informa, 2011; Levin, 2012; Eholzer, 2013) and consolidated feeds (CFTC-SEC, 2010; Nanex, 2010c); (v) delays and dropouts in the transmission of any information due to lost or corrupted messages (Corvil, 2009); and (vi) delays in any messages to or from an execution venue due to excessive message traffic exceeding the capacities of inbound and/or outbound queues; typically when the market is under stress, but also potentially due to deliberate "quote-stuffing" manipulation by traders (Tse et al., 2012).\footnote{Even specialist high-bandwidth interfaces (Eholzer, 2013) may suffer from delays when traffic is excessive. Furthermore, if an exchange provides information about current delays (Eholzer, 2013) in a normal message, this information will itself be delayed.}

The extent of delays can be considerable. For example, order processing times at Eurex are normally $0.2ms - 0.35ms$ but can be delayed by a factor of 10 under normal business conditions and occasionally by a factor of 200 (Eholzer, 2013); and market data reporting from NYSE to the CQS system during the Flash Crash was delayed by $5,000ms - 24,000ms$ (Nanex, 2010c).

Delayed information can substantially affect trading algorithms, since they will make calculations based on incorrect data. For example, a risk-averse market maker operating a two-phase strategy as described above may under-estimate inventory risk and this may lead to a phase-shift from "normal" to "panic" trading. Where all traders are affected by delays then systemic effects such as oscillatory instability may ensue.

Consider the introduction of a delay in the trade-confirmation communications link from the exchange to the traders, and let the confirmations of all executed orders be delayed by an additional $\delta$ time steps where $\delta \in \mathbb{N}$. We model the delayed information on executed orders as four separate components defined as follows:

\[
(dx_{bids_t}, dx_{asks_t}) = (xbids_{(t-\delta)}, xasks_{(t-\delta)}) \\
(dx_{buys_t}, dx_{sells_t}) = (xbuys_{(t-\delta)}, xsells_{(t-\delta)})
\]

This delay is inserted into our model by specifying that the market maker uses the delayed versions rather than the undelayed versions of the executed orders. Equation (1) becomes:

\[
inv_{i,(t+1)} = inv_{i,t} + \psi(i, dx_{bids_t}) + \psi(i, dx_{buys_t}) - \psi(i, dx_{asks_t}) - \psi(i, dx_{sells_t})
\]
The advantages of modelling delays as components are that: (i) delays are made both explicit and precise; (ii) the extent of individual delays may easily be modified with a simple localised change in the model; and (iii) different delays may be inserted at many different points in the market being modelled — for example, if confirmations of sells were delayed twice as much as other orders, we could write \( dx_{sells} = x_{sells}(t-2\delta) \).

4.4. Summary of modelling with recurrence relations

The foregoing equations define our model of the main components of our case study: the exchange, the market makers, and a fundamental seller. We claim that this style of definition, using mutually-recursive recurrence relations, has the advantage that the multiple interactions between the components are made explicit. For example: (i) market maker prices \( bidprice \) and \( askprice \) are coupled to the best bid and best ask prices published by the exchange (Equation (2)); (ii) market maker inventories are coupled to the executed orders \( xbids \), \( xask \), \( xbuys \) and \( xsells \) published by the exchange (Equation (1)); and (iii) the executed orders at the exchange are coupled to the orders received from the traders (Equation (4)).

This equational style provides a highly expressive medium for the description of coupling effects in financial systems with complex dependencies, and we find it to be very useful during the formulation and discussion of hypotheses. It can be used at varying levels of abstraction (it is not necessary for all components to be modelled at the same level of detail) and it supports a wide variety of real behaviours, including information delay.

5. From Coupling to Instability

Here we use our case study to illustrate how we reason about coupling-induced instability in a financial market. First we review the feedback loops in Figure 2, and indicate how instability in trader inventories may lead to instability in market prices. We show that under normal circumstances our market makers are stable, and then we show how instability can be induced by the introduction of an information delay. The remainder of the section shows how we analyse feedback effects and market instability.

For a different case study, e.g. with different pricing functions and strategies, the dynamic interaction model would be different but our reasoning process in relation to coupling, feedback and instability would be the same.
5.1. Feedback loops

Figure 2 gives the bilateral couplings for our case study with one exchange and two market makers, showing how selected components are coupled. The bilateral couplings form chains, and the bolded arrows in the figure show key couplings that turn chains into feedback loops.

Limit order prices are coupled to the best bid and ask prices, which are coupled to the prices of previously-issued limit orders. This forms a feedback loop; either a trader is coupled to him/herself, or a loop covers both traders.

The two bold dashed arrows show the effect of an optional price-banding constraint where order prices are deliberately coupled to the last traded price, thereby creating a feedback loop.

Inventories are coupled to the sizes of trades, and the sizes of trades are coupled to the order sizes, which themselves are coupled to the previous inventories. This creates a dynamic feedback loop (Figure 2) comprising chains of dynamic couplings. We will later show how this feedback loop can induce inventory oscillation.

Market price is an attribute of the executed trades — it is not (absent price-banding) a component of a feedback loop,\footnote{In the presence of price-banding, order prices would be coupled to market price, forming another feedback loop.} though it is coupled to the above feedback loop that connects executed trades to inventories. As the traders’ inventories change, so the limit order book is exposed to changing pressure on traded prices. The bid book comes under price pressure as the total sizes of all sell orders exceeds the total sizes of all bids at the best price, and the ask book comes under pressure as the total sizes of all buys exceeds the total sizes of all asks at the best price. Unbalanced pressure causes the market price to move: balanced pressure leads to liquidity being depleted at the top of both books, more volatile traded prices and increasing spreads.

Figure 5 illustrates the couplings by which unstable inventories might destabilize market price. The feedback loop between inventories and order sizes can be explored in further detail by expanding Equation (1). The size of each execution is the minimum of the resting order size and the executable order size; at each time step the latter is always either $UL$ for the first sell or $-LL$ for the first buy, and the former is given by Equation (6):
The recurrence relation for inventory displays complex feedback: $inv_{i,(t+1)}$ is not only coupled to its previous values at times $t$ and $t-1$ but also to the other trader’s inventory at time $t-1$ ($inv_{j,(t-1)}$). Furthermore, this recurrence relation only holds if trades occur between the two traders — yet, if both start in a stable phase there should be no executable orders and no trades. We shall devote the remainder of this section to the analysis of the dynamic behaviour of our simple market-making strategy: first, to establish its inherent stability, then to demonstrate how it may be destabilized, and finally to explore how two or more such market makers may exhibit self-exciting instability.

5.2. Stability of a single market maker

Here we analyse the dynamic behaviour of a single market maker’s inventory. We establish that under normal conditions if our market maker starts in a stable phase it cannot shift into a panic phase.
A simple algebraic manipulation can be used to establish the stability of the market. From Section 4.2.2, we know that if \( LL + 1 \leq inv_{i,0} \leq UL - 1 \) then our market maker issues only resting limit orders. Consider the extreme case for the maximum achievable inventory — i.e. when at every time step all bids and no asks for the market maker are executed. Furthermore, recall the prerequisites for our case study — that all limit orders are Fill And Kill (so there are no resting bids from before time \( t-1 \)), and that traders issue orders only on even time steps (so \( inv_{i,(t-1)} = inv_{i,(t-2)} \) if \( t \) is even). Now Equation (1) may be explored by expanding terms as follows:

\[
inv_{i,t} = inv_{i,(t-1)} + \psi(i, x\text{bids}_{(t-1)}) + \psi(i, x\text{buys}_{(t-1)}) - \psi(i, x\text{asks}_{(t-1)}) - \psi(i, x\text{sells}_{(t-1)})
\]

\[
= inv_{i,(t-1)} + \psi(i, x\text{bids}_{(t-1)}) - \psi(i, x\text{asks}_{(t-1)}) \quad \because \text{only limit orders}
\]

\[
= inv_{i,(t-1)} + \psi(i, x\text{bids}_{(t-1)}) \quad \because \text{only bids executed}
\]

\[
= inv_{i,(t-1)} + \psi(i, \text{bidbook}_{(t-1)}) \quad \because \text{all bids executed}
\]

\[
= inv_{i,(t-1)} + \psi(i, \text{Bids}_{(t-1)}) \quad \because \text{no old bids, Eq.3}
\]

\[
= inv_{i,(t-1)} + \text{bsize}_{i,(t-2)}
\]

\[
= inv_{i,(t-1)} + \max(0, UL - 1 - inv_{i,(t-2)})
\]

\[
= inv_{i,(t-1)} + \max(0, UL - 1 - inv_{i,(t-1)}) \quad \text{if } t \text{ is even}
\]

\[
= \max(inv_{i,(t-1)}, UL - 1)
\]

Thus \( (UL - 1) \) is a strict, inclusive, upper-bound for the market maker inventory. By a similar argument, \( (LL + 1) \) is a strict, inclusive, lower-bound. We therefore say this market-making algorithm is stable — it will never reach either of its two inventory limits \( UL \) or \( LL \), and therefore will never panic and will never issue aggressive executable orders.

### 5.3. Instability induced by information delay

If a delay were introduced into the market, unknown to the market maker, such that confirmations of all trades were delayed by \( \delta \) time steps, and if there were another trader issuing sells to hit the bids, then we would use the following revised inventory equation, from which (by expanding terms, with the same assumptions as above) we derive a prerequisite for a market maker to shift into panic in such a market.
\[ \text{inv}_{i,t} = \text{inv}_{i,(t-1)} + \psi(i, \text{dxbids}_{(t-1)}) + \psi(i, \text{dbuys}_{(t-1)}) \]
\[ - \psi(i, \text{dasks}_{(t-1)}) - \psi(i, \text{dsells}_{(t-1)}) \]
\[ = \text{inv}_{i,(t-1)} + \psi(i, \text{dxbids}_{(t-1)}) \]
\[ = \text{inv}_{i,(t-1)} + \psi(i, \text{dbids}_{(t-1)}) \]
\[ = \text{inv}_{i,(t-1)} + \psi(i, \text{dxbids}_{(t-1-\delta)}) \quad \because \text{all bids executed} \]
\[ = \text{inv}_{i,(t-1)} + \psi(i, \text{bids}_{(t-1-\delta)}) \quad \because \text{no old bids, Eq. 3} \]
\[ = \text{inv}_{i,(t-1)} + \text{bidsize}_{i,(t-2-\delta)} \]
\[ = \text{inv}_{i,(t-1)} + \max(0, \text{UL} - 1 - \text{inv}_{i,(t-2)}) \]
\[ = \max(\text{inv}_{i,(t-1)}, \text{UL} - 1 + \text{inv}_{i,(t-1)} - \text{inv}_{i,(t-2)}) \]

The trader will phase-shift into panic if \( \text{inv}_t \geq \text{UL} \) and from the above we therefore have the worst-case pre-condition for shifting to panic that:

\[ \text{inv}_{i,(t-1)} > \text{inv}_{i,(t-2-\delta)} \]

5.3.1. Analysing delay (behaviour of shift into panic)

Consider the case where a market maker’s inventory has been stable at value \( \nu \) for some time,\(^{14}\) and then at time \( \tau \) a fundamental buyer enters the market and issues very large sell orders at every time step — sufficient to cause every bid to be executed. Assume \( \delta = 2 \). The previously used algebraic manipulation can be applied, and the changes in inventory for the market maker (which only occur on even timesteps) would be:

<table>
<thead>
<tr>
<th>Time</th>
<th>Inventory</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau )</td>
<td>( \nu )</td>
<td>( \text{xbids}_{(\tau-3)} = { } )</td>
</tr>
<tr>
<td>( \tau + 2 )</td>
<td>( \nu )</td>
<td>( \text{xbids}_{(\tau-1)} = { } )</td>
</tr>
<tr>
<td>( \tau + 4 )</td>
<td>( \max(\nu, (\text{UL} - 1) + \nu - \nu) = \text{UL} - 1 )</td>
<td>( \text{xbids}_{(\tau+1)} \neq { } )</td>
</tr>
<tr>
<td>( \tau + 6 )</td>
<td>( \max(\text{UL} - 1, (\text{UL} - 1) + (\text{UL} - 1) - \nu) &gt; \text{UL} - 1 )</td>
<td>( \nu &lt; \text{UL} - 1 )</td>
</tr>
</tbody>
</table>

Thus, the market maker’s inventory hits or exceeds the limit \( \text{UL} \) at time step \( \tau + 6 \), at which point the market maker shifts into a panic phase and issues executable orders to offload the excess inventory.

---

\(^{13}\)Since traders issue orders only on even time steps, this is equivalent to \( \text{inv}_{i,(t-2)} > \text{inv}_{i,(t-2-\delta)} \) if \( t \) is even.

\(^{14}\)Here, \( \nu \) can take any value \( \nu < (\text{UL} - 1) \) — in Section 5.4.1 we shall return to this example with \( \nu = 2 - \text{UL} \)
More generally, Figure 6 illustrates how the introduction of a delay affects inventory: without a delay, the inventory asymptotically approaches the limit $UL$ (at a rate that depends on the sizes of its executed bids); whereas if a delay is introduced the inventory initially is unchanged because trade confirmations are buffered in the delay component and the market maker issues more orders based on this unchanged inventory, then the first delayed trade confirmation is received and the inventory increases. The subsequent increase in inventory is linear for the same length of time as the inventory was previously unchanged (because in our case study the bid sizes and consequently the executions are directly linked to current inventory), and then the inventory increases at a slower rate because the confirmed trades result now from bids issued at higher inventories. The inventory then hits or exceeds the limit $UL$ and the market maker shifts into the panic phase.

![Figure 6: Inventory grows with and without delay](image)

5.3.2. Analysing delay (shift back into stable phase)

In the panic phase the market maker will try to return to a stable phase as soon as possible by issuing a sell order. Whether this is possible in one transaction depends on both the extent to which the current inventory exceeds the limit and whether the resting liquidity on the order book is sufficient to fully execute the sell order. To provide such liquidity, thereby permitting the market maker to phase-shift back to a stable phase, we would require the other trader to phase-shift its own behaviour so that it issues bids. In the best case for our case study, $\psi(i, x\text{sell}s_{(t-\delta)}) = UL$ (i.e. the market maker’s sell is completely executed) and we have:

\[
\begin{align*}
\text{inv}_{i,t} &= \text{inv}_{i,(t-1)} - \psi(i, d\text{sell}s_{(t-1)}) \\
&= \text{inv}_{i,(t-1)} - \psi(i, x\text{sell}s_{(t-1-\delta)}) \\
&= \text{inv}_{i,(t-1)} - UL
\end{align*}
\]
If the inventory is too high \((\text{inv}_{i,(t-1)} \geq 2UL)\) or the available liquidity is too small \((\psi(i, x\text{sells}_{(t-1-\delta)}) \leq \text{inv}_{i,(t-1)} - UL)\), the market maker will stay in panic and will keep issuing sell orders until (if satisfied) the current inventory falls below the \(UL\) limit.

With the introduction of a very small delay into the market, an oscillatory phase-shifting of another trader can, via unidirectional coupling (with no feedback), induce an oscillatory phase-shifting behaviour in the market maker. Our equational model illustrates very clearly how this occurs: if the other trader phase-shifts between issuing sells and bids, this leads to a market maker oscillation between a positive-inventory panic phase and a stable phase as shown above, and by contrast if the other trader phase-shifts between issuing buys and asks, this leads to a market maker oscillation between a negative-inventory panic phase and a stable phase.

5.4. Self-exciting feedback with two market makers

Here we analyse a market containing a feedback loop, where two market makers can be induced into a self-exciting oscillation. To establish the feedback loop requires a third trader (a fundamental seller to hit the bids), together with a destabilising scenario such as information delay to send one of the market makers into a panic phase. As soon as one of the market makers shifts into panic (it doesn’t matter which one, but we assume that they do not both panic at the same time), the third trader is no longer needed and exits the market. Having achieved a situation where one market maker is in panic and the other is stable, they are able to trade with each other; one issues an executable order and the other issues resting limit orders.

The market maker in panic will reduce its inventory by trading with the stable market maker, and this will change the inventories of both; since the orders subsequently issued by both are dependent on their inventories, there exists a bi-directional coupling between the two market makers. This creates a feedback loop (involving the two market makers, the exchange and the delay component), and we shall demonstrate how this feedback loop is “self-exciting” in that it needs no other component to continue.

This feedback loop can lead to an infinite oscillatory instability between the two market makers, with each shifting in and out of panic in a synchronised contra-oscillation. At first such carefully choreographed contra-oscillation may appear to be unlikely, but our flow analysis will show how the synchronicity arises naturally out of the equations that describe the market, with the action of one component causing the action of the other component.
5.4.1. Information delay with two market makers

We recreate the delay market described in Section 5.3, but now with two market makers and a fundamental trader, and with a delay $\delta$ in all trade confirmations. As before, it is a precondition that the fundamental trader leaves the market as soon as one market maker is in panic and the other is stable (if both market makers panic at exactly the same time step, there will be no trades — the market will remain inactive and therefore stable).

We assume the additional delay $\delta$ from the exchange is unknown to the traders and they are unaware that their current inventories may subsequently be increased or decreased as the result of trade executions that have occurred but whose confirmations have not yet been received. Consequently, a stable market maker may issue a limit order that, if executed, may cause the previously-panicking market maker to become stable and the previously-stable market maker to enter a panic phase.

Our model facilitates analysis and understanding of the behaviour of this market, since it permits the tracking of individual items of the market state (such as orders and confirmations) at each time step. Table 1 illustrates such detailed flows — this flow analysis demonstrates how a starting market state at time $\tau$ where market maker 2 is in positive panic (inventory $2UL - 2$) and the other is stable (inventory $2 - UL$) can without external impetus move first to a market state where both traders are stable (time $\tau + 4$), then to a state where market maker 2 is stable and market maker 1 is in panic (time $\tau + 6$), and finally back again to both being stable (time $\tau + 10$).

In this example, the delayed transit of one trade confirmation is highlighted by a succession of three grey cells. Different patterns of movement in and out of panic are generated with different starting inventories, except that there is a precondition that the initial stable inventory is not $UL - 1$, since this would result in no bids being issued and therefore no trade (the market would be static and stable).

\[\text{\footnote{In general, we need } \delta \text{ \footnote{pending orders} columns for this kind of flow analysis.}}\]
<table>
<thead>
<tr>
<th>time</th>
<th>inv₁ (t)</th>
<th>inv₂ (t)</th>
<th>orders (size)</th>
<th>zorders</th>
<th>pending xorders</th>
<th>dxorders</th>
<th>inv₁ (t + 1)</th>
<th>inv₂ (t + 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>τ</td>
<td>2-UL</td>
<td>2UL-2</td>
<td>(\theta_{b,1,τ} (2UL-3))</td>
<td>(\theta_{b,1,τ} )</td>
<td>(\theta_{S,2,τ} (UL))</td>
<td>2-UL</td>
<td>2UL-2</td>
<td></td>
</tr>
<tr>
<td>τ + 1</td>
<td>2-UL</td>
<td>2UL-2</td>
<td>(\theta_{b,1,τ+2} (2UL-3))</td>
<td>(\theta_{b,1,τ+2} )</td>
<td>(\theta_{S,2,τ+2} )</td>
<td>2-UL</td>
<td>2UL-2</td>
<td></td>
</tr>
<tr>
<td>τ + 2</td>
<td>2-UL</td>
<td>2UL-2</td>
<td>(\theta_{b,1,τ+4} (UL-3))</td>
<td>(\theta_{b,1,τ+4} )</td>
<td>(\theta_{S,2,τ+2} )</td>
<td>2-UL</td>
<td>2UL-2</td>
<td></td>
</tr>
<tr>
<td>τ + 3</td>
<td>2-UL</td>
<td>2UL-2</td>
<td>(\theta_{b,1,τ+6} (UL))</td>
<td>(\theta_{b,1,τ+6} )</td>
<td>(\theta_{S,2,τ+2} )</td>
<td>(\theta_{b,1,τ+6} )</td>
<td>2-UL</td>
<td></td>
</tr>
<tr>
<td>τ + 4</td>
<td>2 UL-2</td>
<td>2 UL-2</td>
<td>(\theta_{b,1,τ+8} (UL-3))</td>
<td>(\theta_{b,1,τ+8} )</td>
<td>(\theta_{S,2,τ+2} )</td>
<td>(\theta_{b,1,τ+8} )</td>
<td>UL+2  -2</td>
<td></td>
</tr>
<tr>
<td>τ + 5</td>
<td>2 UL-2</td>
<td>2 UL-2</td>
<td>(\theta_{b,1,τ+10} (UL+1))</td>
<td>(\theta_{b,1,τ+10} )</td>
<td>(\theta_{S,2,τ+2} )</td>
<td>(\theta_{b,1,τ+10} )</td>
<td>UL+2 -2</td>
<td></td>
</tr>
<tr>
<td>τ + 6</td>
<td>UL+2 -2</td>
<td>UL+2 -2</td>
<td>(\theta_{S,1,τ+6} (UL))</td>
<td>(\theta_{S,1,τ+6} )</td>
<td>(\theta_{b,2,τ+6} )</td>
<td>(\theta_{S,1,τ+6} )</td>
<td>UL+2 -2</td>
<td></td>
</tr>
<tr>
<td>τ + 7</td>
<td>UL+2 -2</td>
<td>UL+2 -2</td>
<td>(\theta_{S,1,τ+8} (UL-3))</td>
<td>(\theta_{S,1,τ+8} )</td>
<td>(\theta_{b,2,τ+8} )</td>
<td>(\theta_{S,1,τ+8} )</td>
<td>UL+2 -2</td>
<td></td>
</tr>
<tr>
<td>τ + 8</td>
<td>UL+2 -2</td>
<td>UL+2 -2</td>
<td>(\theta_{S,1,τ+10} (UL+1))</td>
<td>(\theta_{S,1,τ+10} )</td>
<td>(\theta_{b,2,τ+10} )</td>
<td>(\theta_{S,1,τ+10} )</td>
<td>UL+2 -2</td>
<td></td>
</tr>
<tr>
<td>τ + 9</td>
<td>UL+2 -2</td>
<td>UL+2 -2</td>
<td>(\theta_{S,1,τ+12} (2UL-3))</td>
<td>(\theta_{S,1,τ+12} )</td>
<td>(\theta_{b,2,τ+12} )</td>
<td>(\theta_{S,1,τ+12} )</td>
<td>UL+2 -2</td>
<td></td>
</tr>
<tr>
<td>τ + 10</td>
<td>2 UL-2</td>
<td>2 UL-2</td>
<td>(\theta_{b,1,τ+14} (UL-3))</td>
<td>(\theta_{b,1,τ+14} )</td>
<td>(\theta_{S,2,τ+12} )</td>
<td>(\theta_{b,1,τ+14} )</td>
<td>UL+2 -2</td>
<td></td>
</tr>
<tr>
<td>τ + 11</td>
<td>2 UL-2</td>
<td>2 UL-2</td>
<td>(\theta_{b,1,τ+16} (UL))</td>
<td>(\theta_{b,1,τ+16} )</td>
<td>(\theta_{S,2,τ+12} )</td>
<td>(\theta_{b,1,τ+16} )</td>
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<td></td>
</tr>
<tr>
<td>τ + 12</td>
<td>2-UL</td>
<td>2UL-2</td>
<td>(\theta_{b,1,τ+18} (2UL-3))</td>
<td>(\theta_{b,1,τ+18} )</td>
<td>(\theta_{S,2,τ+12} )</td>
<td>(\theta_{b,1,τ+18} )</td>
<td>2-UL  2UL-2</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Inventory and flow analysis for two panicking HFT market makers: delay \(\delta = 2\); time \(\tau\) is even; HFTs orders are issued on even timesteps and executed on odd timesteps. Negative panics (and executable buy orders) never occur, so asks are not shown. Columns 4 to 7 give: orders issued (\(orders\)); trades executed (\(xorders\)); trade confirmations sent but not yet received (\(pending\ \(xorders\)); and confirmations received (\(dxorders\)). Orders are denoted by \(\theta_b\) (bid) and \(\theta_S\) (sell), followed by the size; panic inventories are bolded. One confirmation flow is highlighted in grey. The rows at times \(\tau\) and \(\tau + 12\) are identical and after time \(\tau + 12\) the market infinitely repeats the flows and inventories from times \(\tau + 1\) to \(\tau + 12\), with both HFTs oscillating in and out of panic.
5.5. Infinite oscillation

The alternate phase-shifting illustrated in Table 1 is due to the bidirectional coupling between the two market makers, and this can lead to an infinite oscillation where two market makers trade with each other indefinitely. This is highly unusual for market makers, who make a loss on each filled executable order\textsuperscript{16} — this behaviour is not motivated by any economic imperative but is an artefact of the unintentional dynamic coupling between the two automated strategies. Our analysis demonstrates that an infinite oscillation is theoretically possible by detecting the case where the market state repeats itself — in particular, the repetition of a sub-state consisting of the two inventories and the outstanding trades whose confirmations have not yet been delivered. This is illustrated in Table 1 at times $\tau$ and $\tau + 12$.

The inventory and flow analysis in Table 1 provides a detailed understanding of how such oscillations may occur, and we have found flow analysis especially helpful in understanding markets with delays. The resulting changes in inventory for the two market makers is further illustrated in Figure 7 (Left).

Manual algebraic manipulation is appropriate for markets involving relatively few instances of coupling, and flow analysis helps to explore market behaviour in great detail over a short timescale. Although it is possible to automate algebraic manipulation using a symbolic algebra application, we have found numerical simulation to be more helpful for modelling and analysing the behaviour of complex feedback markets over longer timescales; we view numerical simulation as an important component of hypothesis formulation, to assist in clarifying hypotheses and the consequences that ensue from the logic embodied in a given hypothesis given certain initial conditions.

We have built a numerical simulator that visualizes our model by animating all its underlying equations through time (as mentioned in Section 4.1). This allows us to monitor time-varying interactions between different components. Our simulator also allows us to expand our recurrence relations to be substantially more complex and to encompass a much greater range of real behaviour, such as randomised order arrival times at the exchange, the execution of crossed bids and asks, and markets with a large number of heterogeneous market makers using different order pricing and sizing functions, with different inventory thresholds. We are therefore able to explore

\textsuperscript{16}In a flat market, they gain the spread on executions of pairs of bid and ask limit orders, but lose the spread on an executable order.
the effect on market instability of different order arrival times, and we are able to demonstrate that the emergent market instability is not dependent on a particular choice of market-making strategy (e.g. it is not simply due to resonance between several identical algorithms).

5.5.1. The phase-shift into panic

A necessary precondition for oscillatory instability is that at least one market maker should be in a panic state. We have previously demonstrated how a fundamental trader can provoke a market maker into a panic state if there is an information delay, and Figure 7 (Right) shows this in simulation for four heterogeneous market makers with randomised order arrival times, leading to a phase-shifting oscillation.

In this example, the four market makers have different inventory limits and different order pricing and sizing functions. For example, resting order sizes are chosen randomly in a range bounded by 0 and a maximum given by Equation (6). They are all initially in a stable state within the inventory limits. However, a fundamental seller (not shown in the figure) who issues a fixed number of executable orders in the first few timesteps can lead one market maker to panic, and then the other market makers.

Figure 7: Left: oscillatory instability with two homogeneous traders moving in and out of panic in contra-correlation. Right: oscillating inventories of four coupled heterogeneous traders in a delayed market — inventories are initially stable, but the market makers are provoked into panic by a single fundamental seller (not shown) issuing sells at the start of the simulation; this seller stops once panic has been provoked. In both figures the shaded zone indicates the stable region of the trader with the smallest inventory limit.

5.5.2. Infinite paired coupling

Figure 8 shows the dynamic inventories of five homogeneous market makers when a market exhibits a minimal information delay of one time step, and
when order arrival times at the exchange are randomised at each time step. For this example, two of the traders start trading from a positive-inventory panic state, another two traders start from a negative-inventory panic state, and the last trader starts with an inventory of zero.

![Inventory Changes](image)

Figure 8: Inventory changes for paired coupling in a market with five homogeneous market makers. The shaded zone is a stable state zone within the algorithms’ inventory limits.

Figure 8 shows the simulation for the first 100 time steps. In roughly the first 25 steps all market makers trade among themselves causing periodic jumps to the panic state and back (due to information delay). These jumps are undesired because the executable orders typically incur financial loss; the market makers therefore try to avoid those jumps by restraining their resting orders when their inventories approach the limits $UL$ and $LL$. In the remaining 75 steps three out of five market makers manage to stabilize their inventories near the limit $UL$. At an inventory of exactly $UL - 1$ they do not issue any bids, and if there are no delayed executions in the pipeline they cannot phase-shift into a positive-inventory panic. However, the other two market makers remain coupled in a feedback loop and continue to trade with each other (similar to Figure 7 (Left)). This leads to an infinite oscillation, where the two market makers repeatedly exchange the same inventory. This could create a continuously false impression of market liquidity.

5.6. How inventory oscillations affect market price

Figure 5 shows how market prices are coupled to the previously described inventory feedback loop, and we have demonstrated how even a very small information delay can trigger that feedback loop to create an oscillating instability in the market maker inventories. What we have not yet demonstrated
is the extent to which inventory instability can affect market price — i.e. the strength of the coupling relationship between inventories and prices.

In Section 4.2.1 we presented Equation (5) to specify how the matching engine “walks the book” in order to fill an executable order, and Figure 3 illustrated how, given a particular distribution of limit orders on the book, a large total size of executable orders is more likely (than a small size) to deplete the top price level on the book and cause a jump in execution price.

The effects on market price are subtle; different distributions of starting inventories lead to different distributions of orders on bidbook and askbook and therefore different probabilities that a particular executable order will cause a price jump. However, in our simple case study we found that a coarse measure of the pressure from large executable orders overwhelming the liquidity on the book can be used as a good “rough guide” to changes in price — it causes prices to change within a single time step, and changes the basic parameters (e.g. best bid and best ask) that drive the pricing functions.

Our coarse measure (for which we make no general claims) subtracts the pressure on resting bids from the pressure on resting asks, and we call this “Net Liquidity Pressure”; if its value is mostly positive we predict rising prices, and if it is mostly negative we predict falling prices.\(^{17}\) In our case study all orders are Fill And Kill, and this simplifies the definitions enormously:\(^{18}\)

\[
Net \text{ Liquidity Pressure}_t = \frac{\sum_i \psi(i, \text{Buys}_t)}{1 + \sum_i \psi(i, \text{Asks}_t)} - \frac{\sum_i \psi(i, \text{Sells}_t)}{1 + \sum_i \psi(i, \text{Bids}_t)}
\]

Figure 9 illustrates the price impact associated with coupling-induced inventory oscillations. The results of two numerical simulations are shown, with graphs set out in two rows — the top row is an example oscillation causing market price to rise, and the bottom row is an example causing price to drop. In each row there are three graphs showing, from left to right, the market maker inventories, the Net Liquidity Pressure, and the market price.

Each simulation comprises a market with an exchange and five heterogeneous market makers (with different inventory limits, different pricing and sizing functions, and where messages to the exchange are randomised at each time step), and a delay in trade confirmations of just one time step ($\delta = 1$).

\(^{17}\)Menkveld (2013) observes that prices are negatively correlated with HFT inventories.

\(^{18}\)The “1+” in the denominator addresses the case where there is no resting liquidity.
In both cases, we assume that prior to the start of the simulation at least one market maker has been induced to panic, and that the fundamental trader has now withdrawn from the market. Thus, whatever happens to the price during these simulations is not due to any fundamental trading — it can only be due to the trading between the market makers themselves.

For the upper simulation, two market makers start with inventories in negative panic and the rest have zero inventories: for the lower simulation, two market makers start with inventories in positive panic and the rest have zero inventories. In both cases, the inventories are highly unstable with repeated phase switches into and out of panic (both positive and negative panics). The net liquidity pressure for the upper simulation is mostly positive, and the rightmost graph shows that market prices rise by about 50% in 500 time steps (equivalent to about 100 ms); the net liquidity pressure for the lower simulation is mostly negative, and the rightmost graph shows market prices dropping by about 40% over the same timescale.

![Graphs showing inventory, liquidity pressure, and market prices](image)

Figure 9: Coupling-induced heterogeneous inventory oscillation, net liquidity pressure, and market prices for two simulations (upper row and lower row). UL and LL are the limits for the trader with the largest limits and UL = -LL for all traders.

The results of Figure 9 indicate that if market makers are induced to trade amongst themselves while other traders exit from the market, then a rapid and appreciable impact on price (up or down) is theoretically possible. Finally, Figure 10 recreates the price behaviour during the Flash Crash of 2010, using public data wherever possible (CFTC-SEC, 2010; Kirilenko
et al., 2010; Nanex, 2010a). Our simulation uses known factors such as the net HFT inventory, total contracts traded within the selected 3 minutes, and the reported mixture of HFT and Opportunistic traders, but there is insufficient public data available for a detailed model and other factors must be assumed or estimated.

Even with very limited public data, our model of dynamic coupling and feedback provides a reasonable approximation to the key price dynamics. Figure 10 illustrates that it is possible to use a low-level dynamic interaction model during hypothesis formulation for understanding real events.

Mimicking the price movements of the Flash Crash is not new (e.g. Padrick et al. (2012)), but our approach has the benefit that it is amenable to formal analysis. We have given some examples of analysis in this article, and we are developing more sophisticated techniques.

6. Conclusion

We have demonstrated how coupling between trading algorithms (especially HFTs) can destabilize markets, and have introduced a new technique for modelling dynamic interaction at varying levels of abstraction. Our case study has shown how unexpected latency and feedback may trigger instability as an unintentional emergent behaviour.

The concept of “coupling” (including static, dynamic and time-dependent coupling) has been defined as a bilateral behavioural dependency between subsystems of a market, where a “subsystem” has been defined inductively to be a single component or an entity comprising other subsystems. We have then defined feedback loops in terms of cyclic chains of couplings, and these
definitions underlie our ability to describe a wide variety of market feedback behaviour at multiple levels of detail, and specifically our ability to model dynamic interaction at the level of the market microstructure.

We have introduced a general framework for modelling dynamic interaction and feedback, where recurrence relations in discrete time are used to express the precise nature of bilateral couplings. Our dynamic interaction models are at a sufficiently low level to express and reason about mechanistic causality, yet are highly flexible in that different parts of the model can be at different levels of detail. The framework also supports the precise expression of communication latencies. We have demonstrated how such models can be used during hypothesis formulation and can be analysed to provide understanding of the causes and triggers of feedback and prerequisites for instability. We have also shown how we use numerical simulation to track the time-varying value of a specific variable such as market price, based on a set of starting conditions and a set of recurrence relations to describe a given market; this provides a further way to analyse the feedback dynamics of a particular model, and we have used this to show how low-level instability in the microstructure of a market can cause high-level instability such as crashes and spikes in market price.

We have explored unexpected latency ("delay") as an example trigger for feedback instability; this has been illustrated with a case study using simple, stable, HFT market makers with inventory constraints in an order-book market. We have explained how dynamic coupling between the HFTs (via the order book) leads to a feedback loop, and how delays can then induce these stable algorithms into an oscillatory instability, phase-shifting with precisely anti-correlated synchrony into and out of inventory panic ("hot potato" trading). We have also shown how coupling-induced feedback between HFTs can be self-exciting — in the absence of other effects, it can lead to a theoretically infinite instability. These effects are induced by the size of delay relative to the frequency of trading; thus, because short delays occur much more frequently than long delays, HFTs are more likely to suffer from these effects than low-frequency traders. In broader terms, our analysis suggests that instability can arise as an unintentional emergent behaviour of markets; i.e. it arises not as a consequence of algorithm complexity or predatory behaviour, but instead as a result of transitive interaction effects. Such emergent instability can arise for a wide range of heterogeneous algorithms with differing order-pricing and order-sizing functions, and is considerably more complex than a simple "resonance" effect. Although we would not expect feedback
loops to cause major market instability during equilibrium trading (due to the large mix of strategies (Hasbrouck and Saar, 2013) and because trades within a feedback loop would be outnumbered by other trades), we do expect feedback to become dominant at times of market breakdown when there are fewer traders, and fewer and more correlated trades.

We believe that the concept of feedback as a cyclic chain of bilateral couplings is essential to understanding emergent instability from stable components. Further, we find that the creation of dynamic interaction models based on recurrence relations is an extremely helpful technique in exploring feedback dynamics, to be used alongside other methods during hypothesis formulation. We have not demonstrated how large-scale dynamic interaction models can be constructed and analysed, and clearly there are important issues still to be resolved such as determining how to analyse a very large market model to determine whether (and how many) feedback loops exist, to compare the relative importance or strength of different feedback loops, and how likely a given market model is to suffer from feedback-induced instability.

Although our case study focuses on oscillation arising from the interaction of HFT market makers, we suspect that many previously observed feedback loops (Danielsson et al., 2012; Zigrand et al., 2012) may also be modelled and analysed using our general framework. Our work may therefore help to understand previously unexplained sources of volatility in financial markets; it may also have implications for models of pricing and market impact, since we demonstrate that traders do not necessarily have independence of action and such models might need to account for unexpected coupling with other traders.

From a practitioner perspective, our dynamic interaction models may help to understand how algorithms and markets could be re-engineered to improve stability. Since even stable algorithms may be subject to dynamic feedback, traders might now decide to test their algorithms for vulnerability to common modes of feedback instability; execution venues might now decide to offer deterministic latency to improve stability, or to monitor feedback effects and provide enhanced information to subscribers; and regulators might decide to use feedback models to help anticipate the efficacy and consequences of proposed regulation — especially during periods of disequilibrium, when regulatory control can be particularly important.
Appendix A. Definitions of functions

$\psi()$

The function $\psi()$ is applied to a sequence of orders $x$ and sums the sizes of all those orders with trader identifier $i$. We use the notation for sequences defined in Section 4.2.1, and we model each order as a quadruplet $(a, b, c, d)$ containing the type of order (a), the size of the order (b), the price of a limit order (c) and the trader identifier (d). The function $\psi()$ is defined as follows:

$$
\psi(i, x) = \begin{cases} 
\emptyset & \text{if } (x = \emptyset) \\
b + \psi(i, r) & \text{if } (x = (a, b, c, d) : r) \text{ and } (d = i) \\
\psi(i, r) & \text{if } (x = (a, b, c, d) : r) \text{ and } (d \neq i)
\end{cases}
$$

$\text{bidprice()}$ and $\text{askprice()}$

$\text{bidprice()}$ and $\text{askprice()}$ calculate the prices of resting limit orders. The prices are varied linearly according to the current inventory (the aim is that inventory should be zero-reverting). The functions each take the same three arguments — the best bid, the best ask, and the inventory. The bid price is greatest when the inventory is smallest (we set $\text{bidprice} = \text{midprice} - 1$ when inventory is $LL + 1$), and the ask price is lowest when inventory is highest (we set $\text{askprice} = \text{midprice} + 1$ when inventory is $UL - 1$). We set $\text{bidprice} = \text{midprice} - 1 - \zeta$ when inventory is $UL - 1$ and $\text{askprice} = \text{midprice} + 1 + \zeta$ when inventory is $LL + 1$, where $\zeta$ is arbitrarily chosen (e.g. we use half the CME price band), and we ensure prices do not become negative.

$$
\text{bidprice}(bb, ba, inv) = \max(0, ((ba + bb)/2) - 1 - \zeta \times (1 - \frac{UL-1-inv}{UL-LL-2}))
$$

$$
\text{askprice}(bb, ba, inv) = \max(0, ((ba + bb)/2) + 1 + \zeta \times (\frac{UL-1-inv}{UL-LL-2}))
$$

$insertask()$ and $insertbid()$

These functions insert a sequence of new orders (argument $z$) into an orderbook sequence (argument $x$). The orderbook must be sorted to ensure price-time ordering: the first order has the lowest price for $askbook$ and the highest price for $bidbook$. We use the notation introduced in Section 4.2.1 for sequences. As explained above, orders are quadruplets — e.g.
(type, size, price, id). The definition below is for $\text{insertask}$(): the definition for $\text{insertbid}$() is identical except that the relational tests are reversed.

$$
\text{insertask}(x, z) = \begin{cases} 
  x & \text{if } (z = \{\}) \\
  \text{insertask}(\{a\}, y) & \text{if } (x = \{\}) \\
  \text{insertask}((\tau, \sigma, \pi, i) : x, y) & \text{if } (x = (d, e, f, g) : q) \\
  (d, e, f, g) : (\text{insertask}(q, z)) & \text{if } (z = (\tau, \sigma, \pi, i) : y) \\
\end{cases}
$$

$\text{match()}$

The function $\text{match()}$ takes a sequence of limit orders ($l$) and a sequence of market orders ($m$), and returns a triple containing (i) a revised sequence of limit orders (after executed orders have been deleted, and partial executions amended), (ii) a sequence of executed limit orders, and (iii) a sequence of executed market orders. We either denote an order by a single letter “$x$” or by a quadruplet “$(a, b, c, d)$” to access its components. The definition (which discards unmatched market orders) follows the structure of Equation 5.

$$
\text{match}(l, m) = \begin{cases} 
  (\{\}, \{\}, \{\}) & \text{if } (l = \{\}) \\
  (l, \{\}, \{\}) & \text{if } (m = \{\}) \\
  (i, (a, b, c, d) : j, (e, f, g, h) : k) & \text{if } (l = (a, b, c, d) : q) \\
  & \text{and } (m = (e, f, g, h) : z) \\
  & \text{and } f < b \\
  & \text{and } (i, j, k) = \text{match}((a, b - f, c, d) : q, z) \\
  (i, (a, b, c, d) : j, (e, b, g, h) : k) & \text{if } (l = (a, b, c, d) : q) \\
  & \text{and } (m = (e, f, g, h) : z) \\
  & \text{and } f > b \\
  & \text{and } (i, j, k) = \text{match}(q, (e, f - b, g, h) : z) \\
  (i, (a, b, c, d) : j, (e, b, g, h) : k) & \text{if } (l = (a, b, c, d) : q) \\
  & \text{and } (m = (e, f, g, h) : z) \\
  & \text{and } f = b \\
  & \text{and } (i, j, k) = \text{match}(q, z) \\
\end{cases}
$$

References
References


